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# Approximation of the Hilbert Transform on the real semiaxis using Laguerre zeros

M.C. De Bonis<sup>a,\*</sup>, B. Della Vecchia<sup>b</sup>, G. Mastroianni<sup>a</sup><sup>a</sup>*Dipartimento di Matematica, Università della Basilicata, C.da Macchia Romana, 85100 Potenza, Italy*<sup>b</sup>*Dipartimento di Matematica, Istituto G. Castelnuovo, Università di Roma La Sapienza, P.le Aldo Moro 2,  
00185 Roma, Italy*

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## Abstract

The authors propose two new algorithms for the computation of Cauchy principal value integrals on the real semiaxis. The proposed quadrature rules use zeros of Laguerre polynomials. Theoretical error estimates are proved and some numerical examples are showed. © 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The Hilbert Transform

$$H(G, t) = \int_0^{\infty} \frac{G(x)}{x - t} dx, \quad t > 0,$$

where the integral is understood in the Cauchy principal value sense, appears in several mathematical problems and is the main part of singular integral equations on  $(0, +\infty)$ . Therefore, the approximation and/or the numerical evaluation of  $H(G, t)$  is of interest. For the finite interval integration several papers have been appeared in the last decade and many algorithms have been proposed. The interested reader can see for example [2,5,6,12,13,20] and the references given therein. For the unbounded interval the corresponding literature is poor. When the integration interval is the whole real line and

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\* Corresponding author.

*E-mail addresses:* [mdebonis@pta.unibas.it](mailto:mdebonis@pta.unibas.it) (M.C. De Bonis), [dellavecchia@iamna.iam.na.cnr.it](mailto:dellavecchia@iamna.iam.na.cnr.it) (B. Della Vecchia), [mastroianni@unibas.it](mailto:mastroianni@unibas.it) (G. Mastroianni).

the density function  $G$  is piecewise analytic with exponential decay, the SINC method [1] gives satisfactory results. For wider classes of functions (for example the Sobolev spaces) the authors in [7] proposed recently some efficient algorithms based on the zeros of Hermite polynomials.

In this paper we want to study the numerical approximation of  $H(G, t)$ . Since  $G(x) = [e^x G(x)]e^{-x} := f(x)e^{-x}$ , we can use polynomial approximations of  $f$  based on the zeros of Laguerre polynomials. Therefore, in the sequel we will consider the more general integral

$$H(fw_\alpha, t) = \int_0^\infty f(x) \frac{w_\alpha(x)}{x-t} dx,$$

$w_\alpha(x) = x^\alpha e^{-x}$ ,  $\alpha \geq 0$ ,  $t > 0$ , for  $f$  belonging to uniform weighted Sobolev type spaces. For the finite interval case, two procedures are used. The first one consists in replacing  $f$  by a Lagrange interpolation polynomial. The second one subtracts the singularity and applies an ordinary Gauss quadrature rule using an additional algorithm to control the term of the quadrature sum containing the knot closest to the singularity. For example, in the case of the integral

$$\int_{-1}^1 \frac{f(x)}{x-t} v^{\alpha, \beta}(x) dx, \quad v^{\alpha, \beta}(x) = (1-x)^\alpha (1+x)^\beta,$$

the mentioned algorithm is based on the fact that, in every closed interval of  $(-1, 1)$ , the distance of two consecutive zeros of  $p_m^{\alpha, \beta}(x)p_{m+1}^{\alpha, \beta}(x)$  ( $p_k^{\alpha, \beta}$   $k$ th Jacobi polynomial) is of the order of  $1/m$ . In the case of Laguerre polynomials a similar property holds only in intervals of type  $[0, 4\theta m]$ ,  $0 < \theta < 1$  (see the proof of Lemma 2.1). Moreover, because of the “bad” distribution of the largest Laguerre zeros, the Gauss–Laguerre formula does not give a good approximation of the corresponding integral (see for example [15]). According to the above considerations we have to modify the procedures used in the interval  $(-1, 1)$  to the case of infinity intervals.

In this paper, we show that, by using “truncated” Gaussian formulas and “truncated” Lagrange interpolating polynomials, the procedures for finite intervals can give efficient and cheap algorithms.

The idea of “truncated” Gaussian formulas was first used in [16]. Here, we give an elementary theoretical motivation and a natural extension of such a procedure (see Section 2).

The paper is organized as follows: in Section 2, we give some results useful in Section 3 and also in different contexts. In Section 3, we give the main results which are proved in Section 4. Finally in Section 5, we show some numerical experiments.

## 2. Preliminary results and notations

### 2.1. Orthonormal polynomials

In the following  $\mathcal{C}$  denotes a positive constant which may assume different values in different formulas. In the sequel,  $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$  means that  $\mathcal{C}$  is independent of  $a, b, \dots$ . Moreover, if  $A, B > 0$  are quantities depending on some parameters, we write  $A \sim B$ , iff there exist two positive constants  $M_1, M_2$ , independent of the parameters of  $A$  and  $B$ , such that

$$M_1 \leq \left(\frac{A}{B}\right)^{\pm 1} \leq M_2.$$

Let  $\{p_m(w_\alpha)\}_{m \in \mathbb{N}}$  be the sequence of the Laguerre orthonormal polynomials with positive leading coefficient, i.e.

$$p_m(w_\alpha) = \gamma_m(w_\alpha)x^m + \cdots, \quad \gamma_m(w_\alpha) > 0,$$

$$\int_{\mathbb{R}^+} p_m(w_\alpha, x) p_n(w_\alpha, x) w_\alpha(x) dx = \delta_{m,n}.$$

The zeros  $x_k := x_{m,k}(w_\alpha)$ ,  $k = 1, \dots, m$ , of  $p_m(w_\alpha)$  satisfy

$$\frac{\mathcal{C}}{m} < x_1 < \cdots < x_m = 4m + 2\alpha + 2 - \mathcal{C}(4m)^{1/3} \quad (2.1)$$

and, uniformly in  $k$  and  $m$  [23]

$$x_k \sim \frac{k^2}{m}, \quad k = 1, \dots, m. \quad (2.2)$$

The distance between two consecutive zeros is estimated as follows:

$$\Delta x_k := x_{k+1} - x_k \sim \sqrt{\frac{x_k}{4m - x_k}}, \quad k = 0, \dots, m, \quad x_0 = 0, \quad x_{m+1} = 4m \quad (2.3)$$

and

$$\Delta x_k \sim \Delta x_{k \pm 1}, \quad k = 0, \dots, m,$$

uniformly with respect to  $m$  and  $k$ . Equivalences (2.3) can be found in [11,14,21].

The following lemma is new and will be useful in the sequel.

**Lemma 2.1.** *Let  $x_{m+1,k}$ ,  $k = 1, \dots, m+1$ , be the zeros of  $p_{m+1}(w_\alpha)$ . Then if  $x_{m+1,k+1}$ ,  $x_{m,k} \in (0, 4\theta m)$ , with  $0 < \theta < 1$  fixed, we have*

$$\frac{1}{x_{m+1,k+1} - x_{m,k}} \leq \mathcal{C} \sqrt{\frac{m}{x_k}}, \quad k = 1, \dots, m, \quad (2.4)$$

where  $\mathcal{C}$  is independent of  $m$  and  $k$ .

Now we recall the well-known Gauss quadrature rule:

$$\int_0^\infty f(x) w_\alpha(x) dx = \sum_{k=1}^m \lambda_k(w_\alpha) f(x_k) \quad \forall f \in \mathbb{P}_{2m-1}, \quad (2.5)$$

where  $x_k$ ,  $k = 1, \dots, m$ , are the Laguerre zeros and  $\lambda_k(w_\alpha)$ ,  $k = 1, \dots, m$ , are called Christoffel numbers. We have the following estimate:

$$\lambda_k(w_\alpha) \sim w_\alpha(x_k) \Delta x_k \quad (2.6)$$

[17], where the constants in “ $\sim$ ” are independent of  $m$  and  $k$ .

## 2.2. Functional spaces

In the sequel we consider the following set of continuous functions on  $\mathbb{R}^+ = (0, +\infty)$  ( $f \in C^0(\mathbb{R}^+)$ ):

$$W_0 := W_0(w_\alpha) := \left\{ f \in C^0(\mathbb{R}^+): \lim_{x \rightarrow 0} |(f w_\alpha)(x)| = 0 = \lim_{x \rightarrow \infty} |(f w_\alpha)(x)| \right\}, \quad \alpha \geq 0$$

with the norm

$$\|f\|_{W_0} := \|fw_\alpha\| := \max_{x \geq 0} |(fw_\alpha)(x)|.$$

The above two limit relations are crucial for the polynomial approximation of  $f \in W_0$ , otherwise the Weierstrass theorem is not true in  $W_0$ . If  $\alpha = 0$  (i.e.  $w_0(x) = e^{-x}$ ) we assume  $f$  continuous in every interval  $[0, b]$ ,  $b < \infty$ , with

$$\lim_{x \rightarrow \infty} |f(x)e^{-x}| = 0$$

and we use the previous norm. In the sequel we will use the following notation:

$$\|fw_\alpha\|_{[a,b]} := \max_{x \in [a,b]} |(fw_\alpha)(x)|.$$

Moreover, we denote by  $W_r = W_r(w_\alpha)$ ,  $r \geq 1$ , the Sobolev-type space defined by

$$W_r := \{f \in W_0 : \|f^{(r)}\varphi^r w_\alpha\| < \infty\}, \quad \varphi(x) = \sqrt{x},$$

equipped with the norm

$$\|f\|_{W_r} := \|fw_\alpha\| + \|f^{(r)}\varphi^r w_\alpha\|.$$

Now we recall that, for all polynomials  $P_m$  of degree  $m = 1, 2, \dots$  and for every  $\delta \in \mathbb{R}^+$ , the following inequality:

$$\max_{x \geq 2/\lambda(m+\alpha+1)(1+\delta)} |P_m x^\alpha e^{-\lambda x}| \leq \mathcal{C} e^{-Am} \max_{0 \leq x \leq 2/\lambda(m+\alpha+1)} |P_m x^\alpha e^{-\lambda x}|, \quad \lambda > 0, \quad (2.7)$$

holds, where the constants  $\mathcal{C}$  and  $A$  depend on  $\delta$  but not on  $m$  and  $P_m$ .

Estimate (2.7) can be deduced easily from [10].

If we want to use the zeros  $x_1 < \dots < x_m$  of the  $m$ th Laguerre polynomial  $p_m(w_\alpha)$  to compute  $H(w_\alpha f, t)$ , the following results will be useful.

For every fixed  $t > 0$  choose  $\vartheta \in (0, 1)$  such that  $t + 1 \leq 4\vartheta m$ . Then let  $x_j$ ,  $j = j(m)$  be the zero of  $p_m(w_\alpha)$  defined by

$$x_j := \min_{k=1, \dots, m} \{x_k \geq 4\vartheta m\}.$$

If  $\Psi \in C^\infty$  is an arbitrary nondecreasing function such that

$$\Psi(x) := \begin{cases} 0, & x \leq 0, \\ 1, & x \geq 1, \end{cases}$$

we construct the function

$$f_j(x) := f_{j(m)}(x) := f(x) - \Psi\left(\frac{x - x_j}{x_{j+1} - x_j}\right) f(x) := f(x)(1 - \Psi_j(x)). \quad (2.8)$$

From (2.8) it follows that  $f_j(x) = f(x)$  if  $x \leq x_j$ ,  $f_j(x) = 0$  if  $x \geq x_{j+1}$  and in addition  $f_j \in W_0$  if  $f \in W_0$ . Now denote by  $E_M(f)_{w_\alpha} = \inf_{P \in \mathbb{P}_M} \|(f - P)w_\alpha\|$ ,  $f \in W_0$ , the best uniform weighted approximation error of the function  $f$ . We recall that if  $f \in W_r$ , then the Favard inequality (see [8])

$$E_M(f)_{w_\alpha} \leq \frac{\mathcal{C}}{\sqrt{M}} E_{M-1}(f')_{w_{\alpha+1/2}} \leq \frac{\mathcal{C}}{(\sqrt{M})^r} \|f^{(r)}\varphi^r w_\alpha\| \quad (2.9)$$

holds.

Then we can state the following:

**Proposition 2.1.** For all function  $f \in W_0$ , we have

$$\|(f - f_j)w_\alpha\| \leq \mathcal{C}[E_M(f)_{w_\alpha} + e^{-bm}\|fw_\alpha\|] \quad (2.10)$$

and

$$E_m(f_j)_{w_\alpha} \leq \mathcal{C}[E_M(f)_{w_\alpha} + e^{-bm}\|fw_\alpha\|], \quad (2.11)$$

where  $M \sim m$  and  $b, \mathcal{C}$  are positive constants independent of  $m, f$  and  $f_j$ .

From Proposition 2.1 we deduce two facts which will be useful to construct our approximations of  $H(w_\alpha f, t)$ .

**Proposition 2.2.** For all functions  $f \in W_r(w_\alpha)$  and for all  $t \in (0, 4\theta m - 1)$ , we have

$$|H(w_\alpha[f - f_j], t)| \leq \mathcal{C} \left[ \frac{\|f^{(r)}\varphi^r w_\alpha\|}{(\sqrt{m})^r} \log m + e^{-Am}\|fw_\alpha\| \right],$$

where  $\mathcal{C}$  and  $A$  are positive constants independent of  $m$  and  $f$ .

It is easy to prove that if

$$|f(x)w_\alpha(x)| \leq \mathcal{C}\|fw_\alpha\|e^{-\lambda x}, \quad x > x_0, \quad \lambda > 0, \quad \mathcal{C} \neq \mathcal{C}(f),$$

then, for  $f \in W_0$  and some  $x_0 > 0$ , we get

$$|H(w_\alpha[f - f_j], t)| \leq \mathcal{C}e^{-4\lambda\vartheta m}\|fw_\alpha\|,$$

where  $t + 1 < 4\vartheta m$  and  $\mathcal{C}$  is independent of  $m$  and  $f$ .

Now denote by  $L_{m+1}(w_\gamma, f)$  the Lagrange polynomial interpolating the function  $f$  at the zeros of  $(4m - x)p_m(w_\gamma, x)$ . Letting  $u_\beta(x) = x^\beta e^{-x/2}$ ,  $-1 < \beta < 1$ , denote by  $W_r(u_\beta)$ ,  $r \geq 0$ , the corresponding Sobolev space. In [17] the authors proved that the estimate

$$\|f - L_{m+1}(w_\gamma, f)\|_{u_\beta} \leq \mathcal{C}E_m(f)_{u_\beta} \log m, \quad \mathcal{C} \neq \mathcal{C}(m, f) \quad (2.12)$$

holds true for every  $f \in W_0(u_\beta)$  if and only if the parameters  $\beta, \gamma > -1$  satisfy the relation

$$2\beta - \frac{5}{2} \leq \gamma \leq 2\beta - \frac{1}{2}. \quad (2.13)$$

Notice that the extra point  $4m$  is necessary for the validity of (2.12). If we do not use such point the  $\log m$  factor in (2.12) is replaced by  $m^{1/6}$ . Now if  $x_j := x_{j,m}(w_\gamma) := \min_k \{x_k(w_\gamma) \geq 4\vartheta m\}$ ,  $\vartheta \in (0, 1)$  and  $f_j(x) := f(x) - f(x)\Psi((x - x_j)/(x_{j+1} - x_j))$ , we have the truncated formula

$$L_{m+1}(w_\gamma, f_j, x) = \sum_{k=1}^j \frac{4m - x}{4m - x_k} \frac{p_m(w_\gamma, x)}{p'_m(w_\gamma, x_k)} \frac{f(x_k)}{x - x_k}.$$

In [18] the following proposition has been proved:

**Proposition 2.3.** For every  $f \in W_0(u_\beta)$ , we have

$$\|f - L_{m+1}(w_\gamma, f_j)\|_{u_\beta} \leq \mathcal{C}\{E_M(f)_{u_\beta} \log m + e^{-Am}\|fu_\beta\|\},$$

with  $M \sim m$  and  $A$  and  $\mathcal{C}$  independent of  $m$  and  $f$ , if and only if (2.13) holds true.

We note that, by (2.12), the choice of  $0 < \theta < 1$  realizes only an economy in computation and not an improvement of the interpolatory process.

### 3. An algorithm for the computation of $H(fw_\alpha, t)$

The previous propositions suggest to propose the following algorithm. For each fixed  $t > 0$ , we choose an integer  $m > 0$  and  $\vartheta \in (0, 1)$  such that  $t + 1 < 4\vartheta m$ . If  $x_k := x_{m,k}(w_\alpha)$ ,  $k = 1, \dots, m$ , are the zeros of  $p_m(w_\alpha)$ , by the previous definitions of  $x_j$ ,  $j = j(m)$  and  $f_j$ , we can write

$$H(w_\alpha f, t) = H(w_\alpha f_j, t) + H(w_\alpha [f - f_j], t) := H(w_\alpha f_j, t) + F_m(t).$$

In virtue of Proposition 2.2, we neglect the term  $F_m(t)$  and approximate  $H(w_\alpha f_j, t)$ .

Since  $f_j(t) = f(t)$ , when  $t + 1 < 4\vartheta m \leq x_j$ , by Gauss formula,  $H(w_\alpha f_j, t)$  can be written for  $x_k \neq t$  as follows:

$$\int_0^\infty f_j(x) \frac{w_\alpha(x)}{x-t} dx = f(t) \left[ \int_0^\infty \frac{w_\alpha(x)}{x-t} dx - \sum_{k=1}^m \frac{\lambda_k(w_\alpha)}{x_k - t} \right] + \sum_{k=1}^j \frac{f(x_k)}{x_k - t} \lambda_k(w_\alpha) + e_m(f_j, t), \quad (3.1)$$

where  $e_m(f_j, t)$  is the remainder term.

Now letting

$$\begin{aligned} A_m(t) &:= \int_0^\infty \frac{w_\alpha(x)}{x-t} dx - \sum_{k=1}^m \frac{\lambda_k(w_\alpha)}{x_k - t}, \\ \Phi_m(f, t) &:= f(t)A_m(t) + \sum_{k=1}^j \frac{f(x_k)}{x_k - t} \lambda_k(w_\alpha), \\ \rho_m(f, t) &:= e_m(f_j, t) + F_m(t), \end{aligned} \quad (3.2)$$

we can write

$$\int_0^\infty f(x) \frac{w_\alpha(x)}{x-t} dx = \Phi_m(f, t) + \rho_m(f, t),$$

which holds for  $x_k \neq t$  obviously.

First, we remark that the second summation in  $\Phi_m(f, t)$  has only  $j$  terms and, for a suitable choice of  $\vartheta$ ,  $j$  may have order  $m/2$ . Hence we avoid the computation of  $m/2$  evaluations of the function with the relative products  $\lambda_k(w_\alpha)f(x_k)$ ,  $k = j+1, \dots, m$ .

If  $|x_k - t|$  is not “too small”, the computation of  $A_m(t)$  is not difficult.

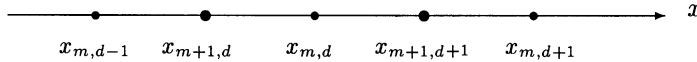
Unfortunately the relation  $t \neq x_k$ ,  $k = 1, \dots, m$ , is not always valid. But even when  $t \neq x_k$ ,  $k = 1, \dots, m$ , the point  $t$  could be too close to one of the Laguerre zeros and this produces numerical instability. On the other hand the only term in (3.2) causing numerical instability is

$$\frac{\lambda_d(w_\alpha)}{x_d - t},$$

where  $x_d$  is a zero closest to  $t$ . Following an argument in [19,4], we now introduce an algorithm in order to control such term.

For every fixed  $t$ , choose  $m_0 = m_0(t) \in \mathbb{N}$  such that for  $m \geq m_0$ , we have  $x_{m,d} \leq t \leq x_{m,d+1}$  for some  $d \in \{1, 2, \dots, m-1\}$ .

Moreover, because of the interlacing properties of the zeros  $x_{m+1,k}$ ,  $k = 1, \dots, m+1$ , of  $p_{m+1}(w_\alpha)$ , we have



Thus, two cases are possible:

(a)  $x_{m+1,d+1} \leq t \leq x_{m,d+1}$

or

(b)  $x_{m,d} \leq t \leq x_{m+1,d+1}$ .

In case (a),

if  $t < (x_{m+1,d+1} + x_{m,d+1})/2$ , then we use the quadrature rule  $\Phi_m(f)$ ;

if  $t \geq (x_{m+1,d+1} + x_{m,d+1})/2$ , then we use the quadrature rule  $\Phi_{m+1}(f)$ .

Similarly in case (b).

Thus, for every fixed  $t$ , we have defined the numerical sequence  $\{\Phi_{m^*}(f, t)\}$ ,  $m^* \in \{m, m+1\}$ .

Moreover, the algorithm for the choice of  $m^*$  is based on Lemma 2.1 and it assures that a knot of  $p_{m^*}(w_\alpha)$  closest to  $t$ , is sufficiently far from  $t$ . Infact, for  $t+1 < 4\theta m$ ,  $0 < \theta < 1$  fixed, there follows:

$$|x_{m^*,d} - t| \geq \mathcal{C} \sqrt{\frac{x_d}{m}}.$$

In particular, if for any  $m$ , we choose

$$t = t_k = \frac{x_k + x_{k+1}}{2}, \quad k = 1, \dots, m-1,$$

then we can evaluate  $H(fw_\alpha, t_k)$  with the required accuracy and we can reconstruct the function  $H(fw_\alpha, t)$ , in the interval  $(0, 4\theta m)$ , by means of suitable interpolating splines.

The next theorem deals with the convergence of the numerical sequence  $\{\Phi_{m^*}(f, t)\}$ .

**Theorem 3.1.** *Let  $t > 0$ ,  $m \in \mathbb{N}$  and  $0 < \theta < 1$ . Then we have*

$$\sup_{0 < t \leq 4m\theta - 1} |\Phi_{m^*}(f, t)| \leq \mathcal{C} \|fw_\alpha\| \log m, \quad f \in W_0, \quad (3.3)$$

and for all  $f \in W_r$  we get

$$|\rho_{m^*}(f, t)| = |H(fw_\alpha, t) - \Phi_{m^*}(f, t)| \leq \mathcal{C} \left\{ \frac{\|f^{(r)}\| \varphi^r w_\alpha}{(\sqrt{m})^r} \log m + e^{-Am} \|fw_\alpha\| \right\}, \quad (3.4)$$

where  $\mathcal{C}$  and  $A$  are positive constant independent of  $m$ ,  $t$  and  $f$ .

Theorem 3.1 ensures that, apart of the  $\log m$  factor, quadrature formula  $\Phi_{m^*}(f, t)$  is stable and converges to  $H(w_\alpha f, t)$  like the best polynomial approximation.

However, Theorem 3.1 holds under the condition  $(t+1)/4\theta < m$ ,  $\vartheta \in (0, 1)$ . Hence when  $t$  is “large”, then the computation of the products  $\lambda_k(w_\alpha)f(x_k)$  is expensive i.e. if  $t = 10\,000$  and  $\theta = \frac{1}{4}$  then  $m \geq 10\,000$ . If it happens, we choose  $m$  and  $\vartheta$  such that  $t > x_{j+1} + 1$ . Consequently  $f_j(x)$ ,

$f_j(x)/(x-t)$  and  $f(x)$  are similarly smooth. Then applying the ordinary Gauss formula to the function  $G_j(x) = f_j(x)/(x-t)$  we get

$$H(w_\alpha f, t) = \sum_{k=1}^j \lambda_k(w_\alpha) \frac{f(x_k)}{x_k - t} + e_m^*(f, t), \quad (3.5)$$

where

$$e_m^*(f, t) = F_m(t) + R_m(G_j)$$

with  $R_m(G_j)$  being the error of the Gauss quadrature formula relative to the function  $G_j$  and  $F_m(t)$  as above.

Obviously, the above quadrature formula is stable (apart of the  $\log m$  factor) and the remainder term can be estimated as follows:

$$|\rho_m^*(f, t)| \leq \mathcal{C} \left\{ \frac{1}{(\sqrt{m})^r} \|G_j^{(r)} \varphi^r w_\alpha\| + e^{-Am} \|f w_\alpha\| \right\}, \quad t < 4\vartheta m - 1, \quad (3.6)$$

where  $\mathcal{C}$  and  $A$  are positive constant independent of  $m$ ,  $t$  and  $f$ .

Finally by using formulas (3.1) and (3.5) we obtain an efficient procedure to compute  $H(w_\alpha f, t)$  with  $t > 0$  in a wide range.

#### 4. Product formulas

In this section, we want to find an approximation of  $H(w_\alpha f, t)$  by replacing the function  $f$  by an algebraic polynomial. We recall that the only known convergent polynomial procedure in  $W_0$  is the sequence of means of the la Vallée–Poussin [22]. However such procedure cannot be used in practice because it requires the analytic computation of the Fourier coefficients. On the other hand, if we approximate such coefficients by a quadrature rule, the de la Vallée–Poussin operator is transformed into a discrete operator whose behaviour is unknown. Then by Proposition 2.3, it seems natural to replace the function  $f_j$  by its Lagrange interpolating polynomial.

Following notations in Section 2, we consider the integral  $H(u_\beta f, t)$ ,  $u_\beta = x^\beta e^{-x/2}$ . We introduce the weight  $u_\beta = x^\beta e^{-x/2}$  only for homogeneity with notations of Proposition 2.3. By dilation, it is possible to obtain, analogous results using the weight  $w_\beta = x^\beta e^{-x}$  and the interpolating polynomial  $L_{m+1}(w_\beta^2, f)$  (here the extra point is  $2m$ ).

If  $\gamma$  satisfies condition (2.13), then consider the sequence  $\{p_m(w_\gamma)\}$ , the zeros  $x_k = x_{m,k}(w_\gamma)$ ,  $k = 1, \dots, m$ , of  $p_m(w_\gamma)$ , the index  $j$  defined by  $x_j = \min\{x_{m,k} \geq 4\vartheta m\}$  and the function  $f_j = f - f\Psi_j$  defined as above. Then we approximate  $H(u_\beta f, t)$  by  $H_m(f, t) := H(u_\beta L_{m+1}(w_\gamma, f_j), t)$ ,  $t > 0$ , where  $L_{m+1}(w_\gamma, f_j)$  is the Lagrange polynomial interpolating  $f_j$  at the zeros of  $p_m(w_\gamma, x)(4m - x)$ . Then  $H_m(f, t)$  can be written as

$$H_m(f, t) = \sum_{k=1}^j \frac{f(x_k)}{4m - x_k} A_k(t), \quad (4.1)$$

where

$$A_k(t) := \int_0^\infty (4m - x) \ell_k(w_\gamma, x) \frac{u_\beta(x)}{x - t} dx$$



with  $x_k := x_{m,k}(w_\gamma)$  and

$$\ell_k(w_\gamma, x) := \frac{p_m(w_\gamma, x)}{p'_m(w_\gamma, x_k)(x - x_k)} = \lambda_k(w_\gamma) \sum_{i=0}^{m-1} p_i(w_\gamma, x_k) p_i(w_\gamma, x).$$

Then

$$A_k(t) = \lambda_k(w_\gamma) \sum_{i=0}^{m-1} p_i(w_\gamma, x_k) \int_0^\infty (4m - x) p_i(w_\gamma, x) \frac{u_\beta(x)}{x - t} dx$$

and the computation of the last integrals is not difficult even if expensive. Now for the sequence  $\{H_m(f)\}_m$  we state the following.

**Theorem 4.1.** *Let  $H_m(f) := H(L_{m+1}(w_\gamma, f_j)u_\beta)$ , with  $\gamma$  satisfying condition (2.13). Then, for all  $f \in W_r$ , we have*

$$\sup_{t+1 \leq 4\theta m} |H(fu_\beta, t) - H_m(f, t)| \leq \mathcal{C} \frac{\|f\|_{W_r}}{(\sqrt{m})^r} \log^2 m,$$

where  $0 < \theta < 1$  is fixed and  $\mathcal{C}$  and  $A$  are positive constant independent of  $m$ ,  $t$  and  $f$ .

Theorem 4.1 shows that  $H_m(f)$  is a stable formula (apart of the  $\log^2 m$  factor) and convergent for all  $f \in W_r(w_\alpha)$ ,  $r \geq 1$ , but it costs more than the previous procedure. On the other hand, as mentioned in the introduction,  $H(w_\alpha f)$  often appears as the main part of singular integral equations and, by a collocation method, it can be replaced by  $H_m(f)$ . So we have to compute the values  $f(x_i)$  ( $x_i \leq 4\theta m - 1$ ) by solving a system of linear equations.

## 5. Proofs

We recall some relations which will be useful in the sequel (see [20]):

$$p'_m(w_\alpha, x) = -\sqrt{m} p_{m-1}(w_{\alpha+1}, x) \quad (5.1)$$

and

$$p_m^2(w_\alpha, x) e^{-x} \left(x + \frac{1}{m}\right)^{\alpha+1/2} \sqrt{4m - x + (4m)^{1/3}} \sim \left(\frac{x - x_d}{x_d - x_{d\pm 1}}\right)^2, \quad (5.2)$$

where  $x \in [0, 4m]$  and  $x_d$  is a zero of  $p_m(w_\alpha)$  closest to  $x$ , i.e.

$$|x_d - x| = \min_{k=1, \dots, m} |x - x_k|.$$

Moreover, for any  $x, y \in [\mathcal{C}/m, 4\theta m]$  and  $|x - y| \leq \mathcal{C} \sqrt{x/m}$ , the following equivalence

$$w_\alpha(x) \sim w_\alpha(y) \quad (5.3)$$

holds (see [8, Lemma 4.1]).

**Proof of Lemma 2.1.** Let  $Q_{2m+1}(x) = p_{m+1}(w_\alpha, x)p_m(w_\alpha, x)$ . Since the zeros of  $p_m(w_\alpha)$  interlace with the zeros of  $p_{m+1}(w_\alpha)$ , then  $Q'_{2m+1}(x_{m+1,k+1}) > 0$ ,  $Q'_{2m+1}(x_{m,k}) < 0$  and

$$0 < Q'_{2m+1}(x_{m+1,k+1}) - Q'_{2m+1}(x_{m,k}) = (x_{m+1,k+1} - x_{m,k})Q''_{2m+1}(\xi_k),$$

where  $x_{m,k} < \xi_k < x_{m+1,k+1}$ . Consequently, by (5.1), we get

$$\begin{aligned} & \frac{1}{x_{m+1,k+1} - x_{m,k}} \\ & < \frac{|Q''_{2m+1}(\xi_k)|}{|Q'_{2m+1}(x_{m+1,k+1})|} \\ & = \sqrt{m} \left[ \frac{[p_{m-2}(w_{\alpha+2})p_{m+1}(w_\alpha) + 2p_{m-1}(w_{\alpha+1})p_m(w_{\alpha+1}) + p_m(w_\alpha)p_{m-1}(w_{\alpha+2})](\xi_k)}{[p_m(w_{\alpha+1})p_m(w_\alpha)](x_{m+1,k+1})} \right]. \end{aligned}$$

By (5.2), with  $d = k$  and  $x = \xi_k$ , the numerator is equivalent to

$$\left( \frac{x_{m,k} - \xi_k}{\Delta x_k} \right)^2 \frac{1}{w_\alpha(\xi_k)\xi_k^{3/2}\sqrt{4m - \xi_k}}.$$

By (5.2), with  $d = k$  and  $x = x_{m+1,k+1}$ , the denominator is equivalent to

$$\left( \frac{x_{m+1,k+1} - x_{m,k}}{\Delta x_k} \right)^2 \frac{1}{w_\alpha(x_{m+1,k+1})x_{m+1,k+1}\sqrt{4m - x_{m+1,k+1}}}.$$

Then, using (5.3),  $x_{m,k} < \xi_k < x_{m+1,k+1}$  and  $x_{m+1,k+1}, x_{m,k} \leq 4\theta m$ ,  $0 < \theta < 1$ , we get

$$\frac{1}{x_{m+1,k+1} - x_{m,k}} \sim \left( \frac{x_{m,k} - \xi_k}{x_{m+1,k+1} - x_{m,k}} \right)^2 \sqrt{\frac{m}{x_k}} \leq \mathcal{C} \sqrt{\frac{m}{x_k}},$$

which proves the lemma.  $\square$

**Proof of Proposition 2.1.** Using definition (2.8), we have

$$\|(f - f_j)w_\alpha\| \leq \|fw_\alpha\|_{[4\theta m, \infty)}.$$

Recalling inequality (2.7), with  $\lambda = 1$  and  $\delta = 1$ , and letting  $M = [m\theta/2]$ , with  $m > 2(\alpha + 1)/\theta$ , for  $P_M \in \mathbb{P}_M$ , it results

$$\max_{x \geq 4\theta m \geq 4(M+\alpha+1)} |P_M(x)w_\alpha(x)| \leq \mathcal{C}e^{-bm}\|w_\alpha P_M\|, \quad (5.4)$$

where  $\mathcal{C}$  and  $b$  are positive constant independent of  $M$  and  $P_M$ . Consequently

$$\|fw_\alpha\|_{[4\theta m, \infty)} \leq \|(f - P_M)w_\alpha\| + \|P_Mw_\alpha\|_{[4\theta m, \infty)} \leq \mathcal{C}(\|(f - P_M)w_\alpha\| + e^{-bm}\|P_Mw_\alpha\|).$$

Taking the infimum on  $P_M$ , we get

$$\|fw_\alpha\|_{[4\theta m, \infty)} \leq \mathcal{C}(E_M(f)_{w_\alpha} + e^{-bm}\|fw_\alpha\|) \quad (5.5)$$

and (2.10) follows.

To prove (2.11), using definition (2.8), for an arbitrary  $P_M$ , with  $M = [m\theta/2]$ , we can write

$$f_j(x) - P_M(x) = f(x) - P_M(x) - f(x)\Psi\left(\frac{x - x_j}{x_{j+1} - x_j}\right)$$

and

$$\|(f_j - P_M)w_\alpha\| \leq \|(f - P_M)w_\alpha\| + \|fw_\alpha\|_{[4\theta m, \infty)}.$$

Taking the infimum on  $P_M$  and using (5.5), we get

$$E_m(f_j)_{w_\alpha} \leq E_M(f_j)_{w_\alpha} \leq \mathcal{C}(E_M(f))_{w_\alpha} + e^{-bm}\|fw_\alpha\|$$

and the proof is complete.  $\square$

**Proof of Proposition 2.2.** For  $t + 1 \leq 4\theta m$ ,  $0 < \theta < 1$ , we have the following decomposition:

$$|H([f - f_j]w_\alpha, t)| \leq \int_{x_j}^{x_{j+1}} \frac{|f(x)w_\alpha(x)|}{x - t} dx + \left| \int_{x_{j+1}}^{\infty} \frac{f(x)w_\alpha(x)}{x - t} dx \right| := A_1 + A_2. \quad (5.6)$$

We have

$$A_1 \leq \|fw_\alpha\|_{[x_j, x_{j+1}]} \left( \int_{x_j}^{x_{j+1}} \frac{dx}{x - t} \right) \leq \mathcal{C}\|fw_\alpha\|_{[4\theta m, \infty)}. \quad (5.7)$$

To estimate  $A_2$ , we can write

$$A_2 \leq \left| \int_{x_{j+1}}^{\infty} f(x)x^{\alpha-1} e^{-x} dx \right| + t \left| \int_{x_{j+1}}^{\infty} f(x) \frac{w_\alpha(x)}{x(x - t)} dx \right| = A'_2 + A''_2. \quad (5.8)$$

Integrating by part, we get

$$\begin{aligned} A'_2 &= \left| \frac{(fw_\alpha)(x_{j+1})}{x_{j+1}} + (\alpha - 1) \int_{x_{j+1}}^{\infty} f(x) \frac{w_\alpha(x)}{x^2} dx + \int_{x_{j+1}}^{\infty} f'(x) \sqrt{x} w_\alpha(x) \frac{dx}{x^{3/2}} \right| \\ &\leq \mathcal{C} \left( \|fw_\alpha\|_{[4\theta m, \infty)} + \frac{\|f' \varphi w_\alpha\|_{[4\theta m, \infty)}}{\sqrt{m}} \right). \end{aligned} \quad (5.9)$$

On the other hand, we have

$$\begin{aligned} A''_2 &\leq \|fw_\alpha\|_{[4\theta m, \infty)} t \int_{x_{j+1}}^{\infty} \frac{dx}{x(x - t)} = \|fw_\alpha\|_{[4\theta m, \infty)} \log \frac{x_{j+1}}{x_{j+1} - t} \\ &\leq \mathcal{C}\|fw_\alpha\|_{[4\theta m, \infty)} \log m. \end{aligned} \quad (5.10)$$

Combining (5.9) and (5.10) with (5.8), it follows

$$A_2 = \left| \int_{x_{j+1}}^{\infty} \frac{f(x)w_\alpha(x)}{x - t} dx \right| \leq \mathcal{C} \left( \|fw_\alpha\|_{[4\theta m, \infty)} \log m + \frac{\|f' \varphi w_\alpha\|_{[4\theta m, \infty)}}{\sqrt{m}} \right). \quad (5.11)$$

Thus, using (5.7) and (5.11) into (5.6), we deduce

$$|H([f - f_j]w_\alpha, t)| \leq \mathcal{C} \left( \|fw_\alpha\|_{[4\theta m, \infty)} \log m + \frac{\|f' \varphi w_\alpha\|_{[4\theta m, \infty)}}{\sqrt{m}} \right)$$

and applying (5.5), we get

$$|H([f - f_j]w_\alpha, t)| \leq \mathcal{C}[E_M(f)_{w_\alpha} + e^{-bm}\|fw_\alpha\|] \log m \\ + \frac{\|(f - P_M)' \varphi w_\alpha\|_{[4\theta m, \infty)}}{\sqrt{m}} + \frac{\|P_M' \varphi w_\alpha\|_{[4\theta m, \infty)}}{\sqrt{m}},$$

where  $P_M$  is the polynomial of best approximation of  $f$  of degree  $M = [m\theta/2]$ . Now we recall the following inequality (see [20])

$$\frac{\|(f - P_M)' \varphi w_\alpha\|_{[0, \infty)}}{\sqrt{m}} \leq \mathcal{C}E_M(f)_{w_\alpha} + \frac{1}{\sqrt{m}}E_{M-1}(f')_{w_{\alpha+1/2}}. \quad (5.12)$$

Moreover, using (2.7) and then the Bernstein inequality, we have

$$\frac{\|P_M' \varphi w_\alpha\|_{[4\theta m, \infty)}}{\sqrt{m}} \leq \mathcal{C}e^{-bm} \frac{\|P_M' \varphi w_\alpha\|_\infty}{\sqrt{m}} \leq \mathcal{C}e^{-bm} \|P_M w_\alpha\| \leq \mathcal{C}e^{-bm} \|fw_\alpha\|. \quad (5.13)$$

Thus, we get

$$|H([f - f_j]w_\alpha, t)| \leq \mathcal{C} \left( E_M(f)_{w_\alpha} \log m + e^{-bm}\|fw_\alpha\| + \frac{1}{\sqrt{m}}E_{M-1}(f')_{w_{\alpha+1/2}} \right).$$

Finally, by using the Favard inequality

$$E_M(f)_{w_\alpha} \leq \frac{\mathcal{C}}{\sqrt{m}}E_{M-1}(f')_{w_{\alpha+1/2}} \leq \frac{\mathcal{C}}{(\sqrt{m})^r} \|f^{(r)} \varphi^r w_\alpha\|$$

the proposition follows.  $\square$

In order to prove Theorems 3.1 and 4.1 and inequality (3.6) we need the following lemmas.

**Lemma 5.1.** *Let  $w_\alpha(x) = x^\alpha e^{-x}$ ,  $\alpha > 0$ , and let  $\{G_m\}$  be a sequence of functions belonging to  $W_1$ . Then, for arbitrary positive  $a$  and  $a'$  such that  $a' < a$ , we have*

$$\sup_{0 \leq t \leq a'} \left| \int_0^a \frac{(w_\alpha G_m)(x)}{x - t} dx \right| \leq \mathcal{C}_\alpha \left[ \|w_\alpha G_m\|_{[0, a]} \log(am) + \frac{1}{\sqrt{m}} \|G_m' \varphi w_\alpha\|_{[0, a]} \right],$$

where  $\mathcal{C}_\alpha$  depends only on  $\alpha$ .

**Proof.** Let  $t = 0$ . Then

$$\int_0^a G_m(x) x^{\alpha-1} e^{-x} dx = \left\{ \int_0^{1/m} + \int_{1/m}^a \right\} G_m(x) x^{\alpha-1} e^{-x} dx := A_1 + A_2. \quad (5.14)$$

Integrating by part, we have

$$A_1 = \frac{1}{\alpha} (G_m w_\alpha) \left( \frac{1}{m} \right) + \frac{1}{\alpha} \int_0^{1/m} (w_\alpha G_m)(x) dx - \frac{1}{\alpha} \int_0^{1/m} (w_\alpha G_m' \varphi)(x) \frac{dx}{\sqrt{x}}.$$

Consequently

$$|A_1| \leq \left( \frac{1}{\alpha} + \frac{1}{\alpha m} \right) \|w_\alpha G_m\|_{[0, 1/m]} + \frac{2}{\alpha \sqrt{m}} \|G'_m \phi w_\alpha\|_{[0, 1/m]}. \quad (5.15)$$

Moreover

$$|A_2| \leq \int_{1/m}^a |(w_\alpha G_m)(x)| \frac{dx}{x} \leq \|w_\alpha G_m\|_{[1/m, a]} \log(am). \quad (5.16)$$

Using (5.15) and (5.16) in (5.14), we get

$$\left| \int_0^a G_m(x) x^{\alpha-1} e^{-x} dx \right| \leq 2 \|w_\alpha G_m\|_{[0, a]} \log(am) + \frac{2}{\alpha \sqrt{m}} \|G'_m \phi w_\alpha\|_{[0, a]}.$$

Now, we assume  $0 < t \leq 1 < a$  and we use the following decomposition

$$\int_0^a (w_\alpha G_m)(x) \frac{dx}{x-t} = \left\{ \int_0^{2t} + \int_{2t}^a \right\} (w_\alpha G_m)(x) \frac{dx}{x-t} := I_1 + I_2. \quad (5.17)$$

We have

$$\begin{aligned} |I_1| &\leq w_\alpha(t) \left| \int_0^{2t} \frac{G_m(x) - G_m(t)}{x-t} dx \right| + \|w_\alpha G_m\|_{[0, 2t]} \int_0^{2t} \frac{|w_\alpha(x) - w_\alpha(t)|}{w_\alpha(x)|x-t|} dx \\ &\leq \int_0^{2t} \Delta_{u/2\sqrt{t}} G_m(t) \frac{w_\alpha(t)}{u} du + \mathcal{C}_\alpha \|w_\alpha G_m\|_{[0, 2]} \\ &\leq \int_0^{1/\sqrt{m}} \left| \frac{w_\alpha(t)}{u} \int_{t-(u/2)\sqrt{t}}^{t+(u/2)\sqrt{t}} G'_m(y) dy \right| du + \mathcal{C} \|w_\alpha G_m\|_{[0, 2]} \log m + \mathcal{C}_\alpha \|w_\alpha G_m\|_{[0, 2]} \\ &\leq \int_0^{1/\sqrt{m}} \frac{1}{u\sqrt{t}} \int_{t-(u/2)\sqrt{t}}^{t+(u/2)\sqrt{t}} |G'_m(y) \sqrt{y} w_\alpha(y)| dy du + \mathcal{C}_\alpha \|w_\alpha G_m\|_{[0, 2]} \log m \\ &\leq \mathcal{C}_\alpha \left( \frac{1}{\sqrt{m}} \|G'_m \phi w_\alpha\|_{[0, 2]} + \|w_\alpha G_m\|_{[0, 2]} \right) \log m. \end{aligned} \quad (5.18)$$

In order to estimate  $I_2$ , we first assume  $2t < 1/m$ . We write

$$|I_2| \leq \left| \int_{2t}^{1/m} G_m(x) \frac{w_\alpha(x)}{x-t} dx \right| + \left| \int_{1/m}^a G_m(x) \frac{w_\alpha(x)}{x-t} dx \right| := I'_2 + I''_2. \quad (5.19)$$

It is easy to see that

$$I''_2 \leq \|w_\alpha G_m\|_{[1/m, a]} \log ma. \quad (5.20)$$

Moreover, since  $x \geq 2t$  implies  $x-t \geq x/2$ , it results that

$$\begin{aligned} I'_2 &\leq \left| \int_{2t}^{1/m} G_m(x) \frac{w_\alpha(x)}{x} dx \right| + \left| t \int_{2t}^{1/m} G_m(x) \frac{w_\alpha(x)}{x(x-t)} dx \right| \\ &\leq \left| \frac{1}{\alpha} \int_{2t}^{1/m} G_m(x) e^{-x} dx \right| + \left( 2t \int_{2t}^\infty \frac{dx}{x^2} \right) \|w_\alpha G_m\|_{[2t, 1/m]}. \end{aligned}$$

Integrating by part, we deduce

$$\begin{aligned} I'_2 &\leq \mathcal{C}_\alpha \|w_\alpha G_m\|_{[2t, 1/m]} + \frac{1}{m\alpha} \|w_\alpha G_m\|_{[2t, 1/m]} + \frac{1}{\alpha\sqrt{m}} \|G'_m \phi w_\alpha\|_{[2t, 1/m]} \\ &\leq \mathcal{C}_\alpha \left( \frac{1}{m} \|w_\alpha G_m\|_{[2t, 1/m]} + \frac{1}{\sqrt{m}} \|G'_m \phi w_\alpha\|_{[2t, 1/m]} \right). \end{aligned} \quad (5.21)$$

Thus, using (5.20) and (5.21) in (5.19), for  $2t < 1/m$ , we get

$$I_2 \leq \mathcal{C}_\alpha \left( \|w_\alpha G_m\|_{[2t, 1/m]} \log(am) + \frac{1}{\sqrt{m}} \|G'_m \phi w_\alpha\|_{[2t, 1/m]} \right). \quad (5.22)$$

Now, if  $2t \geq 1/m$ , we have

$$I_2 \leq \|w_\alpha G_m\|_{[0, a]} \int_{1/m}^a \frac{dx}{x} = \|w_\alpha G_m\|_{[0, a]} \log(am). \quad (5.23)$$

Then, combining (5.18), (5.22), (5.23) with (5.17), we get

$$\int_0^a (w_\alpha G_m)(x) \frac{dx}{x-t} \leq \mathcal{C}_\alpha \left( \|w_\alpha G_m\|_{[0, a]} \log(am) + \frac{1}{\sqrt{m}} \|G'_m \phi w_\alpha\|_{[0, a]} \right), \quad (5.24)$$

where  $0 \leq t \leq 1$  and  $\mathcal{C}_\alpha$  depends only on  $\alpha$ .

By the decomposition  $[0, t - 1/m] \cup [t - 1/m, t + 1/m] \cup [t + 1/m, a]$ , the case  $1 < t \leq a'$  is easy and we leave the detail to the reader.  $\square$

We recall that if a function  $g$  is such that  $g^{(i)}(x) \geq 0$ ,  $i = 0, 1, \dots, 2m - 1$ ,  $m > 1$ , for  $x \in [0, x_d]$ ,  $d = 2, \dots, m$ , then

$$\sum_{i=1}^{d-1} \lambda_i(w_\alpha) g(x_i) \leq \int_0^{x_d} g(x) w_\alpha(x) dx \leq \sum_{i=1}^d \lambda_i(w_\alpha) g(x_i). \quad (5.25)$$

If  $(-1)^i g^{(i)}(x) \geq 0$ ,  $i = 0, 1, \dots, 2m - 1$ ,  $m > 1$ , for  $x \in [x_d, \infty)$ ,  $d = 1, \dots, m - 1$ , then

$$\sum_{i=d+1}^m \lambda_i(w_\alpha) g(x_i) \leq \int_{x_d}^\infty g(x) w_\alpha(x) dx \leq \sum_{i=d}^m \lambda_i(w_\alpha) g(x_i) \quad (5.26)$$

(see [3, Proof of Lemma 5.1 (b) p. 271–272]).

Letting

$$A_{m^*}(t) = \int_0^\infty \frac{w_\alpha(x)}{x-t} dx - \sum_{\substack{k=1 \\ k \neq d}}^m \frac{\lambda_k(w_\alpha)}{x_k - t},$$

we prove the following:

**Lemma 5.2.** *Let  $0 < t \leq 4\theta m$ , with  $0 < \theta < 1$ , then*

$$A_{m^*}(t) \leq \mathcal{C} w_\alpha(t) \quad (5.27)$$

hold, where  $\mathcal{C}$  is a positive constant independent of  $m$  and  $f$  and  $m^* \in \{m, m+1\}$ .

**Proof.** Consider the case  $x_{d-1} < x_d \leq t < x_{d+1}$ ,  $d \in \{2, \dots, m-1\}$ . By using (5.25) and (5.26), we have

$$A_{m^*}(t) \leq \frac{\lambda_{d-1}(w_\alpha)}{t - x_{d-1}} + \int_{x_{d-1}}^{x_{d+1}} \frac{x^\alpha e^{-x}}{x - t} dx := A + B. \quad (5.28)$$

Since  $t - x_{d-1} \geq (x_d - x_{d-1}) = \Delta x_{d-1}$ , by (2.3), (2.6) and (5.3), we deduce

$$|A| \leq \mathcal{C} w_\alpha(x_{d-1}) \leq \mathcal{C} w_\alpha(t). \quad (5.29)$$

To estimate  $B$ , we note that

$$B = \int_{x_{d-1}}^{x_{d+1}} \frac{w_\alpha(x) - w_\alpha(t)}{x - t} dx + w_\alpha(t) \int_{x_{d-1}}^{x_{d+1}} \frac{dx}{x - t} \leq \mathcal{C} w'_\alpha(\zeta) \Delta x_d + w_\alpha(t) \log \frac{x_{d+1} - t}{t - x_{d-1}}.$$

Since  $x_{d+1} - t \sim t - x_{d-1}$ , using (5.3) it follows

$$B \leq \mathcal{C} w_\alpha(t). \quad (5.30)$$

Thus, replacing (5.29) and (5.30) into (5.28), we prove the lemma.  $\square$

**Proof of Theorem 3.1.** We first prove inequality (3.3). We note that

$$|\Phi_{m^*}(f, t)| \leq |f(t)| |A_{m^*}(t)| + \sum_{k=1}^j \frac{|f(x_k)|}{|x_k - t|} \lambda_k(w_\alpha) := A + B. \quad (5.31)$$

Since  $|x_{m^*,d} - t| \geq \Delta x_{m^*,d}$ , recalling Lemma 5.2 we get

$$A \leq \mathcal{C} \|f w_\alpha\|. \quad (5.32)$$

To estimate  $B$ , by (2.6) we have

$$B \leq \sum_{k=1}^j \frac{\Delta x_k}{|x_k - t|} |f(x_k) w_\alpha(x_k)| \leq \|f w_\alpha\|_{[0, 4\theta m]} \sum_{k=1}^j \frac{\Delta x_k}{|x_k - t|}.$$

Taking into account that  $|x_{m^*,d} - t| \geq \Delta x_{m^*,d}$ , we have

$$\frac{\Delta x_k}{|x_k - t|} \sim \frac{1}{|d - k| + 1}$$

and then

$$B \leq \mathcal{C} \|f w_\alpha\| \log m. \quad (5.33)$$

Combining (5.32) and (5.33) with (5.31), (3.3) follows.

Now we estimate (3.4). We recall that

$$|\rho_{m^*}(f, t)| \leq |e_{m^*}(f_j, t)| + |F_{m^*}(t)|. \quad (5.34)$$

Since the estimate of  $F_{m^*}(t)$  is in Proposition 2.2, it remain to estimate  $e_{m^*}(f_j, t)$  with  $0 < t \leq 4\theta m - 1$ . To this end, we write

$$\begin{aligned} |e_{m^*}(f_j, t)| &\leq \left| \int_0^{4m} \frac{f_j(x) - P_m(x)}{x - t} w_\alpha(x) dx \right| + \left| \int_{4m}^\infty \frac{P_m(x)}{x - t} w_\alpha(x) dx \right| \\ &\quad + |\Phi_{m^*}(f_j - P_m, t)| := I_1 + I_2 + I_3, \end{aligned} \quad (5.35)$$

where  $P_m \in \mathbb{P}_m$  is the polynomial of best approximation of  $f_j$ . To estimate  $I_1$  we use Lemma 5.1 with  $a' = 4\theta m - 1$  and  $a = 4m$  to obtain

$$I_1 \leq \mathcal{C} \left[ \|(f_j - P_m)w_\alpha\| \log m + \frac{\|(f_j - P_m)' \varphi w_\alpha\|}{\sqrt{m}} \right]$$

by which, in virtue of (5.12), it follows

$$I_1 \leq \mathcal{C} \left[ E_m(f_j)_{w_\alpha} \log m + \frac{1}{\sqrt{m}} E_{m-1}(f'_j)_{w_{\alpha+1/2}} \right].$$

Then, applying (2.11) and (2.9), for all  $f \in W_r$  and  $M = [m\theta/2]$ , we deduce

$$\begin{aligned} I_1 &\leq \frac{\mathcal{C}}{(\sqrt{M})^r} E_{M-r}(f^{(r)})_{w_{\alpha+r/2}} \log m + \mathcal{C} e^{-bm} \|f w_\alpha\| \\ &\leq \mathcal{C} \left[ \frac{1}{(\sqrt{m})^r} \|f^{(r)} \varphi^r w_\alpha\| \log m + e^{-bm} \|f w_\alpha\| \right]. \end{aligned} \quad (5.36)$$

To estimate  $I_2$  we use (5.11), with  $f = P_m$ , and we obtain

$$I_2 \leq \mathcal{C} \left[ \|P_m w_\alpha\|_{[4\theta m, \infty)} \log m + \frac{\|P'_m \varphi w_\alpha\|_{[4\theta m, \infty)}}{\sqrt{m}} \right].$$

Using (2.7) and the Bernstein inequality, we deduce

$$I_2 \leq \mathcal{C} e^{-bm} \|f w_\alpha\|. \quad (5.37)$$

Finally, to estimate  $I_3$  we use (3.3), (2.11) and (2.9) to obtain

$$\begin{aligned} I_3 &\leq \mathcal{C} E_m(f_j)_{w_\alpha} \log m \leq \mathcal{C} [E_M(f)_{w_\alpha} + e^{-bm} \|f w_\alpha\|] \log m \\ &\leq \mathcal{C} \left[ \frac{\|f^{(r)} \varphi^r w_\alpha\|}{(\sqrt{m})^r} + e^{-bm} \|f w_\alpha\| \right] \log m. \end{aligned} \quad (5.38)$$

Combining (5.36), (5.37) and (5.38) with (5.35), the proof is complete.  $\square$

**Proof of (3.6).** In [16] has been proved that, for all  $G \in W_r$  it results

$$\begin{aligned} \left| \int_0^\infty G(x) w_\alpha(x) dx - \sum_{k=1}^j \lambda_k(w_\alpha) G(x_k) \right| &\leq \frac{\mathcal{C}}{(\sqrt{m})^r} \int_0^\infty |G^{(r)}(x)| \varphi^r(x) w_\alpha(x) dx \\ &\quad + \mathcal{C} e^{-Am} \int_0^\infty |G(x) w_\alpha(x)| dx. \end{aligned}$$

Letting  $G_j = f_j(x)/(x-t)$ , estimate (3.6) easily follows.  $\square$

**Proof of Theorem 4.1.** We have

$$\begin{aligned} |H(f u_\beta, t) - H_m(f, t)| &= \left| \int_0^\infty \frac{[f - L_{m+1}(w_\gamma, f_j)](x)}{x-t} u_\beta(x) dx \right| \\ &\leq |F_m(t)| + \left| \int_0^\infty \frac{[f_j - L_{m+1}(w_\gamma, f_j)](x)}{x-t} u_\beta(x) dx \right| \\ &:= A + B. \end{aligned} \quad (5.39)$$



Since  $A$  has been estimated in Proposition 2.2, we have to estimate  $B$ . To this end we write

$$B \leq \left| \int_0^{8m} \frac{[f_j - L_{m+1}(w_\gamma, f_j)](x)}{x-t} u_\beta(x) dx \right| + \left| \int_{8m}^\infty \frac{L_{m+1}(w_\gamma, f_j)(x)}{x-t} u_\beta(x) dx \right| \\ := A_1 + A_2. \quad (5.40)$$

To estimate  $A_2$  we use (5.11) with  $f = L_{m+1}(w_\gamma, f_j)$  and with the same condition on  $t$ . Since condition (2.13) is satisfied, we have

$$A_2 \leq \mathcal{C} \left[ \|L_{m+1}(w_\gamma, f_j) u_\beta\|_{[8m, \infty)} \log m + \frac{\|L'_{m+1}(w_\gamma, f_j) \varphi u_\beta\|_{[8m, \infty)}}{\sqrt{m}} \right].$$

Using (2.7), the Bernstein inequality and (2.12),

$$A_2 \leq \mathcal{C} e^{-bm} \|f u_\beta\| \quad (5.41)$$

easily follows, where  $\mathcal{C}$  and  $b$  are independent of  $m$  and  $f$ .

To estimate  $A_1$  we use Lemma 5.1, with  $a' = 4\theta m - 1$  and  $a = 8m$ , and we obtain

$$A_1 \leq \mathcal{C} \left[ \|(f_j - L_{m+1}(w_\gamma, f_j)) u_\beta\| \log m + \frac{\|(f_j - L_{m+1}(w_\gamma, f_j))' \varphi u_\beta\|}{\sqrt{m}} \right] := A'_1 + A''_1. \quad (5.42)$$

By (2.10), Proposition 2.3 and (2.9), we have

$$A'_1 \leq \mathcal{C} \left[ \frac{\|f^{(r)} \varphi^r u_\beta\|}{(\sqrt{M})^r} \log^2 m + e^{-Am} \|f u_\beta\| \right]. \quad (5.43)$$

Moreover, using [20, Corollary 3.7 with  $s = r$  and  $r = 1$ ], it results

$$A''_1 \leq \mathcal{C} \frac{\|f_j^{(r)} \varphi^r u_\beta\|}{(\sqrt{m})^r} \log m.$$

It remains to estimate the norm at the right-hand side. Since  $f_j(x) = 0$  for  $x \geq x_{j+1}$ , we can write

$$\|f_j^{(r)} \varphi^r u_\beta\| \leq \|f^{(r)} \varphi^r u_\beta\|_{[0, x_j]} + \|f_j^{(r)} \varphi^r u_\beta\|_{[x_j, x_{j+1}]}.$$

Letting  $I_j := [x_j, x_{j+1}]$  and  $|I_j| := x_{j+1} - x_j$ , we have

$$f_j^{(r)} = f^{(r)} - \sum_{k=0}^r \binom{r}{k} f^{(k)} \Psi_j^{(r-k)}$$

and

$$|\Psi_j^{(r-k)}(x)| \leq S |I_j|^{k-r},$$

where  $S := \max_i \|\Psi_j^{(i)}\|$  and  $|I_j|^{-1} \sim 1$ . Then, recalling (5.3), it results

$$\|f_j^{(r)} \varphi^r u_\beta\|_{I_j} \sim (\varphi^r u_\beta)(x_j) \|f_j^{(r)}\|_{I_j} \leq \mathcal{C} \|f^{(r)} \varphi^r u_\beta\|_{I_j} + \mathcal{C} (\varphi^r u_\beta)(x_j) \sum_{k=0}^r \binom{r}{k} |I_j|^k \|f^{(k)}\|_{I_j}.$$

Using the following inequality [9, p. 15]:

$$|I_j|^k \|f^{(k)}\|_{I_j} \leq \mathcal{C} [\|f\|_{I_j} + |I_j|^r \|f^{(r)}\|_{I_j}],$$

it follows

$$\|f_j^{(r)} \varphi^r u_\beta\|_{I_j} \leq \mathcal{C}(\|f u_\beta\|_{I_j} + \|f^{(r)} \varphi^r u_\beta\|_{I_j})$$

and then

$$A_1'' \leq \frac{\mathcal{C}}{(\sqrt{m})^r} (\|f u_\beta\| + \|f^{(r)} \varphi^r u_\beta\|) \log m. \quad (5.44)$$

Combining (5.43) and (5.44) with (5.42), we get

$$A_1 \leq \frac{\mathcal{C}}{(\sqrt{m})^r} \|f\|_{W_r} \log^2 m$$

and the proof is complete.  $\square$

## 6. Numerical evaluations

In this section, we show some approximate values for the integral  $H(f w_\alpha, t)$ ,  $t \in \mathbb{R}^+$ , obtained by using the algorithms described in Sections 3 and 4.

The density functions we choose are representatives of the functional spaces (e.g. Sobolev spaces) on which we want to test our method, i.e. we do not exclude the integrals could be better calculated otherwise.

In the following examples we apply our rules taking  $\theta = \frac{1}{4}$  and, as we can see, we obtain the same precision with a number of points  $j$  that is only a little bit more than half of the total

Table 1

$m$	$j$	$\Phi_m$		
		$t = 0.1$	$t = 1.5$	$t = 10$
8	5	1.2593	−0.749	−0.0815
17	10	1.259397	−0.749889	−0.0815154
34	21	1.259397171	−0.749889351	−0.08151545767
68	41	1.25939717184125	−0.749889351885332	−0.081515457671179

Table 2

$m$	$j$	$H_m$		
		$t = 0.1$	$t = 1.5$	$t = 10$
8	5	1.2	−0.7	−0.08
16	10	1.25	−0.749	−0.081
32	19	1.259	−0.749	−0.0815
64	39	1.259397	−0.749889	−0.0815154
128	78	1.259397171	−0.749889351	−0.0815154576
256	156	1.2593971718412	−0.74988935188533	−0.081515457671179

number  $m$ . Since the exact values of the integrals is not known, the results on the last line of our tables are thought to be exact to the number of figures shown. Moreover, in all the tables we have reported only the digits which are correct according to these exact values.

**Example 1.** We want to evaluate the following integral:

$$\int_0^\infty \frac{\cos(\log(1+x))}{x-t} e^{-x} dx.$$

Since the function  $f(x) = \cos(\log(1+x))$  is a very smooth function, we obtain very accurate results. In Table 1, we approximate the above integral by using the quadrature formula (3.2). We can see that, for different values of the parameter  $t$ , we need only 41 points to obtain the machine precision.

In Table 2 we show the corresponding results obtained by using the quadrature formula (4.1).

Table 3

 $\beta = 1$ 

$t = 0.1$			$t = 1.5$			$t = 5$		
$m$	$j$	$\Phi_m$	$m$	$j$	$\Phi_m$	$m$	$j$	$\Phi_m$
17	10	2.12	17	10	−0.8	20	12	−0.36
35	21	2.12	34	21	−0.8	41	25	−0.36
70	43	2.124	68	41	−0.849	83	50	−0.36
140	85	2.1242	137	83	−0.8497	166	101	−0.3639

Table 4

 $\beta = 2$ 

$t = 0.5$			$t = 1.5$			$t = 10$		
$m$	$j$	$\Phi_m$	$m$	$j$	$\Phi_m$	$m$	$j$	$\Phi_m$
21	13	−0.7731	17	10	−0.777	17	10	−0.084
43	26	−0.773199	34	21	−0.77779	35	21	−0.084478
87	53	−0.77319989	68	41	−0.7777965	71	43	−0.0844782
175	106	−0.773199894	137	83	−0.77779651	143	87	−0.08447825

Table 5

 $\beta = 3$ 

$t = 1.5$			$t = 5$			$t = 500$		
$m$	$j$	$\Phi_m$	$m$	$j$	$\Phi_m$	$m$	$j$	$\Phi_m$
17	10	−0.588	20	12	−0.1	16	10	−0.0011
34	21	−0.58811	41	25	−0.129981	32	19	−0.0011790
68	41	−0.5881136	83	50	−0.129981197	64	39	−0.001179098
137	83	−0.588113675254	166	101	−0.1299811973714	128	78	−0.0011790988199

We can note that formula  $H_m$  needs a greater number of points to obtain the machine precision. Moreover, it has a more expensive computational cost than  $\Phi_m$ .

**Example 2.** Now we evaluate the following integral:

$$\int_0^\infty \frac{dx}{(1+x^2)^\beta(x-t)}, \quad \beta = 1, 2, 3.$$

If  $\beta = 1$ , then the function  $f(x) = e^x/(1+x^2) \in W_4$  and the theoretical error goes like  $m^{-2} \log m$ ; if  $\beta = 2$ , then the function  $f(x) = e^x/(1+x^2)^2 \in W_8$  and the theoretical error goes like  $m^{-4} \log m$ ; if  $\beta = 3$ , then the function  $f(x) = e^x/(1+x^2)^3 \in W_{12}$  and the theoretical error goes like  $m^{-6} \log m$ . In Tables 3–5 we can see that the numerical results agree with the theoretical ones.

All the computations were done in double precision arithmetic on a Digital Ultimate Workstation 533 au<sup>2</sup>.

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